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Positivity Bounds in Effective Field Theory

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Abstract

This dissertation is an introduction to the basic concepts of the Effective Field Theory. For a low energy EFT to admit UV- completion, it must satisfy requirements of the underlying full theory: localilty, unitarity, analyticity and Lorentz-invariance. These requirements result in constraints on the scattering amplitudes that must be satisfied for the EFT to be UV-complete. Following the S- program requirements in the forward scattering limit, one obtains positivity constraints on the scattering amplitude. This leads to the positivity bounds on Wilson coefficients, which have a wide range of applications in theoretical and experimental physics. This procedure was explicitly shown for 2- to- 2 scattering of scalar fields, and reviewed for the massive vector field 2- to- 2 scattering. Examples of applications of positivity bounds on coupling constants, such as for the fermionic sector of Standard Model EFT were then discussed.

Acknowledgements

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1 Introduction

The idea of Effective Field Theory (EFT) was pioneered by nobel- prize winning physicists Kenneth Wilson and Steven Weinberg. Kenneth Wilson was working on renormalization group methods in 1960s- 1970s (see, e.g. [1]), exploring methods to describe field theories in the high- energy limit. Weinberg was working on methods to derive empirical quantities in a systematic and effective approach by exploring non- fundamental Lagrangians [2]. Wilson formulated his first prototype of effective theory in 1965 in [3], describing an idea of separating degrees of freedom according to the energy scale, as well as couplings associated with low energy effective models with corresponding cutoff scales, the Wilson coefficients [4].

Effective Field Theory (EFT) now is a powerful tool to describe physics at a given energy scale to a certain accuracy using Quantum Field Theory (QFT) consisting of a finite set of operators. Thus, considering a system at a low energy, one can use EFT to predict low energy observables without specifying details of the physics at high energies, in case it is unknown or difficult to measure (e.g. a field is too heavy to detect). The low and high energy physics are separated by a so- called cut- off scale Λ. This is sometimes referring to the maximum energy achieved in high energy accelerator experiments, or cosmological observations. Although as years go by the highest energy to be probed by colliders gets increased, it is currently impossible to imagine to be able to measure (or experimentally achieve) high energies way above Λ.

Complete theories of the Universe that also include energies above Λ are ultraviolet (UV) theories. These complete UV theories can be approximated in the low energy limit, giving us infrared (IR) theories. For example the Standard Model (SM) in itself should be seen as an EFT, it has been rigorously tested at different accelerators around the globe. However, it is expected that SM is only valid up to some cutoff scale [5]. To systematically parameterize physics beyond the SM, one uses Standard Model Effective Field Theory (SMEFT) [6], [7]. In a sense, each theory can be viewed as an EFT: Quantum Electrodynamics (QED)- relativistic QFT, is an approximation to SM, where all the SM particles have been integrated out except for photons and electrons.

To approach UV- completion of a theory, i.e. ensure that the theory is valid at any energy scale, but is approximated by the low energy limit, one can follow two methods: top- bottom or bottom-up.

In the top- bottom method, one starts with the UV- complete theory that includes both light ($m < \Lambda$) and heavy ($m > \Lambda$) particles. Then, for the range where the light particles predominate, one can integrate out heavy particle fields, leaving only light particles in IR regime, which gives the desired EFT.

One can not always be sure that the theory can be UV- completed, that is when the bottom- up method is preferred: one starts with a low energy model, finds the light particle spectrum, and then builds a UV- completion from it. Generally speaking one imposes assumptions about the fundamental UV theory and as a result derives powerful constraints that have to be satisfied by IR physics (EFTs). In case they cannot be satisfied by the low energy physics one concludes that the UV completion with such initial assumptions is not possible. In the String Theory literature, EFTs which do not admit a UV completion are referred to as part of the swampland [8], [9].

Building an EFT one adds operators in the Lagrangian that are local and preserve symmetries of the theory. However, that leaves a huge freedom on space of allowed (initially unconstrained) coefficients. Physical principles can reduce the freedom of this parameter space. For a low energy EFT to be UV- completed, a number of constraints must be implemented [10]. Assuming that the complete theory must obey the principles of QFT and relativity, namely unitarity, analyticity, locality and Lorentz invariance [11], leads to the requirement that for a scattering event, the associated scattering matrix $\hat{S}$ must be unitary, causal and analytic. This will constrain the number of possible coefficients (couplings) added in the EFT Lagrangian. It will be seen in this project that these constraints lead to positivity bounds of the couplings. These positivity bounds in turn have a wide range of applications: they can be applied in the cosmological EFTs of scalars, vectors and gravity [12] , [13], as well as studies beyond the Standard Model [14]. From the relatively recent works, positivity bounds can be related to the Weak Gravity Conjecture (e.g. see [15]), convexity of charged operators [16], pion scatterings [17].

Effective theory is also used to describe Quantum Chromodynamics (QCD), e.g. Heavy Quark Effective Theory (or HQET) and non -relativistic QCD describe hadrons composite of heavy quarks (bottom and charm) in the low energy region [18], [19]. Chiral perturbation theory describing interactions of pions and nucleons at low momentum, developed in 1960s by Weinberg [20], where one starts from UV- completed full theory however meets difficulties to analytically match onto EFT, with progress on this matching made in [21].

In this project I explore what constraints these properties of $\hat{S}$- matrix lead to, namely unitarity, crossing symmetry, causality and analyticity. I will rederive some theorems like Optical theorem, and will use the results of Schwartz reflection principle and Froissart bound, to calculate the positivity constraints. I will consider an elastic 2-2 scattering of scalar fields. I will start with a theory only including light fields and will rederive the corresponding scattering amplitude $A(s, 0)$ in the forward scattering limit $t = 0$, and by applying the positivity constraint on second derivative of $A(s, 0)$ w.r.t. s, the positivity bound on Wilson coefficient will be obtained. I will then add a heavy field in the theory resulting in new terms in the Lagrangian, this is now the UV- complete theory. By finding equations of motion for the heavy field, and substituting them back into the

Lagrangian, one 'integrates out' this heavy field bringing us back into the IR region with light fields only. Thus, we get back to the initial low energy EFT with only light fields, and see that the coefficients in the UV and EFT Lagrangians can be equated, giving result that the Wilson coefficient must be positive.

I will then review a paper that explores vector field scattering: a more complicated case, due to non- trivial scattering amplitude, as well as complications coming from the polarization of vectors. Although all the cases considered in this dissertation are at the tree- level, there are a number of works on computing scattering amplitudes beyond the tree- level: at loop Feynman diagrams, one recent work example is [22]. In this paper 4- Higgs interactions are considered.

1.1 Brief Review of Scattering Amplitudes

In what follows we will work in the flat metric convention is $\eta_{\mu\nu} = diag(1, -1, -1, -1)$, which is more frequently used in High Energy Physics.

1.1.1 Mandelstam variables

The main focus will be the two- particle → two- particle scattering amplitude. For 2-2 scattering the momentum conservation at the vertex implies $p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$, as depicted in figure 1.

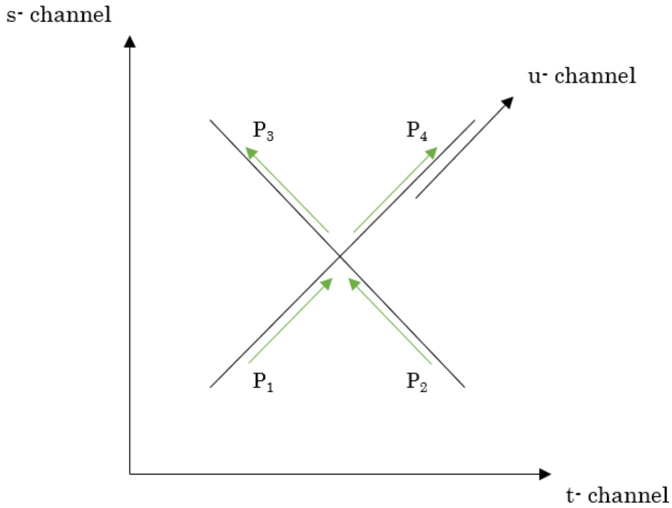

Figure 1: Feynman diagram for 2- 2 scattering at tree level expressed in terms of s-, t- , u- channels.

The on- shell condition for free particles and external legs implies $p^2 = m^2$, for each particle, noting that for internal lines on Feynman diagrams the particles are generally off-shell. For tree- level diagrams, it is convenient to define a set of relativistic invariants encoding energies and momenta of the scattering particles, called the Mandelstam variables:

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_1 - p_3)^2 = (p_4 - p_2)^2 \\ u &= (p_1 - p_4)^2 = (p_3 - p_2)^2, \end{aligned} \tag{1}$$

noting that when reading literature with $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$, expressions for s, t and u will include an opposite sign. Here, $p_i = (E_i, \vec{p_i})$, and $p_i \cdot p_j = p^\mu p_\mu$. From the on- shell condition and momenta conservation law it follows that $s + t + u = \sum_{i=1}^{4} m_i^2 = 4m^2$. It is useful to consider an eikonal approximation, where in the ultra- relativistic case, masses of the scattering particles are so small allowing to write

$$s \approx 2p_1 \cdot p_2, \qquad t \approx 2p_1 \cdot p_3, \qquad u \approx 2p_1 \cdot p_4. \tag{2}$$

The Mandelstam variables correspond to the centre of mass energy of the system.

1.1.2 Scattering Kinematics

In this project the scattering events are assumed to be elastic. That implies that ingoing and outgoing states are indistinguishable: e.g. for 2-2 scattering $m_1 = m_3, m_2 = m_4$. In other words this process refers to no kinematic energy loss: neither of each particle, nor of the system in total. Moreover, an additional constraint can be imposed on the angle of the scattering: forward scattering limit. Assuming the particles scatter at 0 angle implies $t = 0$. Thus, we imposed momentum and energy conservation on the scattering process.

1.1.3 $\hat{S}$- matrix derivation via perturbative QFT

In this section the method of deriving $\hat{S}$ -matrix is explored.
The Hamiltonian as an operator $\hat{H}$ generates time evolution via a unitary operator $\hat{U} = e^{-i\hat{H}\Delta t}$. It can be broken down into a free part $\hat{H}_0^S$ and interaction part $\hat{H}_{int}^S$ in Schrödinger picture as (3)

$$\hat{H}^S = \hat{H}_0^S + \hat{H}_{int}^S. \tag{3}$$

Now we move to the interaction picture, which is a hybrid of the Schrödinger and Heisenberg pictures, in the sense that both operators and states have time dependence and we want to see how they evolve over time. For the operators we use the free evolution operator $\hat{U}_0 = e^{-i\hat{H}_0\Delta t}$ as

$$\hat{H}^I = \hat{U}_0^{-1}\hat{H}^S\hat{U}_0 = \hat{U}_0^{-1}(\hat{H}_0^S + \hat{H}_{int}^S)\hat{U}_0, \tag{4}$$

noticing that if there is no interaction $\hat{H}_{int}^S = 0$, then eqn. (4) simply gives a Hamiltonian in Heisenberg picture. In Schrödinger picture all operators are time- independent, in the interaction picture the Hamiltonian now has time dependence, but only on its interaction part, as the free part commutes with the unitary evolution operator and thus remains time independent:

$$\hat{H}_0^I(t) = \hat{U}_0^{-1}\hat{H}_0^S\hat{U}_0 = \hat{U}_0^{-1}\hat{U}_0\hat{H}_0^S = \hat{H}_0^S \qquad and \qquad \hat{H}_{int}^I(t) = \hat{U}_0^{-1}\hat{H}_{int}^S\hat{U}_0 \neq \hat{H}_{int}^S. \tag{5}$$

The states, on other hand, evolve with full time evolution operator $\hat{U} = e^{-it\hat{H}^S}$ as (6)

$$|\Psi(t_2)\rangle^S = \hat{U}(t_2 - t_1)|\Psi(t_1)\rangle^S. \tag{6}$$

Moving to the interaction picture via $|\Psi(t)\rangle^I = \hat{U}_0^{-1}|\Psi(t)\rangle^S$, we see a state evolves from t_1 to t_2 in the following way:

$$|\Psi(t_2)\rangle^I = \hat{U}_0^{-1}(t_2)\hat{U}(t_2 - t_1)\hat{U}_0(t_1)|\Psi(t_1)\rangle^I = \hat{U}(t_2, t_1)|\Psi(t_1)\rangle^I. \tag{7}$$

Differentiating (7) with respect to time,

$$\frac{d}{dt}(U_I(t,t_0)) = \frac{d}{dt}(e^{itH_0^S}e^{-i(t-t_0)H^S}e^{-it_0H_0^S}) = iH_0^SU_I(t,t_0) - ie^{itH_0^S}(H_0^S + H_{int}^S)e^{-i(t-t_0)H^S}e^{-it_0H_0^S}, \tag{8}$$

gives the Schrödinger evolution equation that has to be solved for $U_I(t,t_0)$:

$$\frac{d}{dt}(U_I(t,t_0)) = -iH_{int}^IU_I(t,t_0). \tag{9}$$

Note the initial condition for evolution operator at t_0: $\hat{U}_I(t_0,t_0) = \mathbb{1}$. Then one can compute $\hat{U}_I$ from (9) using perturbation theory via Dyson expansion of $-i\int dt\hat{H}_I\hat{U}_I$, to get Dyson series of the form:

$$\begin{aligned}\hat{U}_I(t,t_0) &= \mathbb{1} - i\int_{t_0}^{t} dt'\hat{H}_I(t') + (-i)^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\hat{H}_I(t')\hat{H}_I(t'') + ... \\ &= \mathbb{1} - i\int_{t_0}^{t} dt'\hat{H}_I(t') + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt'\int_{t_0}^{t} dt''T(\hat{H}_I(t')\hat{H}_I(t'')) + ... \\ &= Te^{-i\int_{t_0}^{t} dt'\hat{H}_I(t')},\end{aligned} \tag{10}$$

where T is time ordering operator defined so that

$$T\hat{A}(t)\hat{B}(t') = \Theta(t - t')\hat{A}(t)\hat{B}(t') + \Theta(t' - t)\hat{B}(t')\hat{A}(t). \tag{11}$$

Then taking initial state at time t_i $|i\rangle$ its propagation into the final state at time t_f is determined by evolution operator $|f\rangle = \hat{U}_I(t_f,t_i)|i\rangle$ and the amplitude is

$$A = \langle f|\hat{U}_I(t_f,t_i)|i\rangle\ . \tag{12}$$

The $|i\rangle$ and $|f\rangle$ states form a complete basis of Hilbert space at $t = \pm\infty$, respectively. We can compute amplitude A of scattering of incoming m particles with momenta $p_1, p_2, ..., p_m$ in far past $t_i \to -\infty$ into outgoing n particles with momenta $q_1, q_2, ..., q_n$ in far future $t_f \to \infty$ by considering an $\hat{S}$- matrix:

$$A =_{out} \langle q_1, q_2, ..., q_n | p_1, p_2, ..., p_m \rangle_{in} =_{in} \langle q_1, q_2, ..., q_n | \hat{S} | p_1, p_2, ..., p_m \rangle_{in}. \tag{13}$$

Noting that A must be translation invariant to be non-zero, implying $\sum_{i=1}^{m} p_i = \sum_{i=1}^{n} q_i$. Splitting the $\hat{S}$- matrix into free and non- trivial propagation parts $\hat{S} = \mathbb{1} + i\hat{T}$, for initial and final momenta being unequal, we are left with $i\hat{T}$ part, so one defines reduced matrix element $A_{p\to q}$ as

$$_{in}\langle q_1, q_2, ..., q_n | \hat{S} | p_1, p_2, ..., p_m \rangle_{in} =_{in} \langle q_1, q_2, ..., q_n | i\hat{T} | p_1, p_2, ..., p_m \rangle_{in} = (2\pi)^4 \delta^{(4)}(\sum_{p_i}^{m} - \sum_{q_i}^{n}) i A_{p\to q}. \tag{14}$$

Recalling that 1- particle state with momentum p in the far past can be written in terms of the creation operator and the true vacuum Ω as $|p\rangle_{in} = \hat{a}_p^\dagger(-\infty)|\Omega\rangle$, where creation and its conjugate, annihilation, operators obey usual ladder commutation relations. This allows us to construct Fock basis of Hilbert space for multiparticle state $|p_1, ..., p_m\rangle_{in}$ of particles far separated in the far past and similarly for out states. Thus, writing amplitude A in (13) in terms of evolution operator like in (12) and using expression for $U_I(t_f, t_i)$ obtained in (10) one finds:

$$\begin{aligned} A &= \lim_{t\to\infty} \langle q_1, ..., q_n | T(e^{-i\int_{-t}^{t} dt' \hat{H}_I(t')}) | p_1, ... p_m \rangle \\ &= \lim_{t\to\infty} \prod_{i=1}^{n} (2E(\vec{q}_i) \int d^3 y_i e^{i q_\mu^i y_i^\mu}) \prod_{i=1}^{m} (2E(\vec{p}_j) \int d^3 z_j e^{i p_\mu^j z_j^\mu}) \\ &\times \langle \Omega|_t : \hat{\phi}_I(y_1) ... \hat{\phi}_I(y_n) : T(e^{-i\int dt H_I}) : \hat{\phi}_I(z_1) ... \hat{\phi}_I(z_n) : |\Omega\rangle_t \Big|_{y_i^0 = t, z_i^0 = -t}. \end{aligned} \tag{15}$$

Knowing the form of the interacting Hamiltonian $\hat{H}_I$, the exponent can be expanded giving terms of free propagation, and terms of interactions with 1 vertex, 2 vertices, etc., containing correlation functions that can be evaluated using Wick's theorem.

1.2 Construction of the EFT Lagrangian

To compute the scattering amplitude at low energies all the information needed is encoded in the EFT Lagrangian. The method from lectures in [23] will be closely followed alongside [24].
The assumption of locality leads to a separation of scales: short distance Lagrangian coefficients C_i and long- distance matrix elements M_{ij}. Then one can define observables verified by experiments as a product of coefficients and matrix elements $\mathcal{O}_i = \sum_i C_i M_{ij}$. To write down an EFT Lagrangian one has to determine the field content. For example for low energy EFT the Lagrangian will only have light fields, and we must specify their masses, spins and symmetries. In general, in d spacetime dimensions, any Lagrangian density has mass dimension d. Any Lagrangian can be written as a sum of local, gauge and Lorentz invariant operators O_i of dimension D and coefficients c_i of dimension $(d - D)$:

$$\mathcal{L}(x) = \sum_i c_i O_i(x), \tag{16}$$

where each term is of dimension d. Thus we can summarize all possible gauge and Lorentz invariant operators of dimension $D \leq d$, for $d = 4$:

$$1, \phi, \phi^2, \phi^3, \bar{\psi}\psi, \phi^4, \phi\bar{\psi}\psi, D_\mu\phi D^\mu\phi, \bar{\psi}i\not{D}\psi, X_{\mu\nu}^2, \tag{17}$$

where ϕ is a scalar field, ψ is a Dirac (fermion) spinor field, covariant derivative $D_\mu = \partial_\mu + igA_\mu$, and gauge field strength $X_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \ldots$, and all other possible operators are either a combination of the mentioned ones, or vanish during integration over spacetime.
For an EFT Lagrangian the same rules apply as for any other Lagrangian, but now the dimension of allowed operators D does not have to be smaller than d: one can include higher dimension operators due to extra scale Λ^{D-d}, which also ensures that coefficient c is dimensionless:

$$\mathcal{L}_{EFT} = \sum_i \frac{c_i O_i}{\Lambda^{D-d}}. \tag{18}$$

Λ corresponds to the cutoff scale that separates UV and IR regimes. The coefficients c_i are known as Wilson coefficients or Low Energy Constants, which in the 'bottom -up' method are a priori undetermined and unconstrained, and can be fixed by using experimental data. In the 'top - bottom' method these coefficients are determined by matching with factors in front of the computed amplitudes from UV- complete theory. For SMEFT the Lagrangian would be of form:

$$\mathcal{L} = \mathcal{L}_{SM} + \sum_{d>4} \mathcal{L}^{(d)}, \tag{19}$$

where Standard Model Lagrangian is $\sum_i \sum_{d=0}^{4} \frac{c_i O_i^{(d)}}{\Lambda^{d-4}}$ and the second term is a sum of possible Lagrangians containing invariant higher dimension operators [25].

1.3 Crossing Symmetry

In this section it will be shown that the S- matrix satisfies crossing symmetry and the consequences of it. The 4- particle scattering amplitude $A(s,t,u)$ can be expressed as a function of two Mandelstam variables only: s and t, since the on- shell condition gives $s+t+u=4m^2$. Recalling from section 1.1.1: $s=(p_1+p_2)^2=(p_3+p_4)^2, t=(p_1-p_3)^2=(p_4-p_2)^2$. Relabelling particle momenta p_i to i, and anti- particle momenta $-p_i$ to i' one can express 2-2 scattering in s-, u- and, t- channels depicted in 1 as:

$$\begin{aligned} &1+2 \to 3+4, \quad \text{for s- channel} \\ &1+3' \to 2'+4, \quad \text{for t- channel} \\ &1+4' \to 2'+3, \quad \text{for u- channel} \end{aligned} \tag{20}$$

One should define physical regions (often called physical domains) of $s-, t-, u-$ channels corresponding to respective scattering amplitude $A(s,u)$. From the on- shell condition, as well as discussion on analyticity in 1.5 the physical region is defined for $s \geq 4m^2$, which implies $t \leq 0, u \leq 0$, for s-channel, where s is the center of mass energy.

Alternatively, one could consider scattering in u- channel, then the physical region can be analytically continued to $u \geq 4m^2$,$t \leq 0, s \leq 0$. Similarly, for u- channel $t \geq 4m^2$,$u \leq 0, s \leq 0$.

One can also define Mandelstam triangle: the region between the tree physical domains $0 < s,t,u < 4m^2$, shown in figure 2. Mandelstam triangle is the interior region enclosed by lines $t=4M^2, s=(m+M)^2, u=(m+M)^2$, here $m=M$ is taken for simplicity [26].

The Mandelstam triangle describes region with analytic amplitude allowing to analytically extend into each of s-, t- and u- channel domains [27].

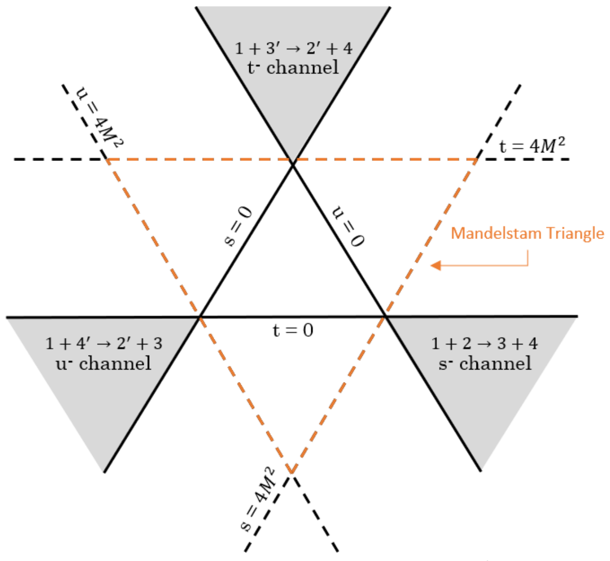

Figure 2: Mandelstam triangle: the region enclosed by lines $t = 4M^2, s = 4M^2, u = 4M^2$. The physical regions of 2-2 scattering in s-, t-, u- channels is highlighted in grey.

The crossing symmetry [28] imply that for 4 scalar particles the amplitudes of s- and u- channels are related by

$$A^{1+2\to3+4}(s,t) = A^{1+4'\to2'+3}(u,t). \tag{21}$$

Analogously, s- and t- channels are related as

$$A^{1+2\to3+4}(s,t) = A^{1+3'\to2'+4}(t,s). \tag{22}$$

For four indistinguishable scalars, the crossing symmetry relation can be simplified to

$$A(s,t) = A(u,t) = A(t,s). \tag{23}$$

This result will be used in the analytical analysis of the amplitude. The crossing symmetry is the fundamental property of S- matrices that ensures relativistic invariance of the theory, as it provides invariance of physical processes under e.g. rotation from s- channel to the t- channel. Another important consequence of crossing symmetry is an additional branch cut in the complex s- plane. In section 1.5 one branch cut is obtained to be starting from $s = (m_i + m_j)^2 = (2m)^2$ (for $m_i = m_j$) towards infinity along the real s axis. The other branch cut must exist at $s = (m_i - m_j)^2 = 0$, for $m_i = m_j$ towards $-\infty$ along real s axis [29]. These expressions are derived for the elastic case, and for the inelastic scatterings there is additional contribution from the intermediate states. It should also be noted that for particles with a spin the exchange of s- and u- channels is not as trivial, and in general is not equivalent to crossing symmetry.

1.4 Unitarity

Time evolution is a unitary process, implying that $\hat{S}$- matrix must be unitary. In this section we will see how this leads to an Optical Theorem which has important results used in deriving positivity bounds.

Assume some initial state $|i\rangle$ transforms into a different state $|i'\rangle$ as $|i'\rangle = \hat{S}|i\rangle$ and similarly the final state $\langle f|$ tranforms into a different state $\langle f'| = \langle f|\hat{S}^\dagger$, then due to the transformation invariance of the expectation values and conservation of the probability,

$$\langle f'|i'\rangle = \langle f|\hat{S}^\dagger \hat{S}|i\rangle \ = \langle f|i\rangle, \quad \forall f, i. \tag{24}$$

We can immediatly conclude unitarity: (24) implies that $\hat{S}^\dagger \hat{S} = \mathbb{1}$. For $\hat{S}$ being the scattering matrix, it would leave the initial and final states untransformed, so the unitarity would be even more trivial. However, this is not the same for non- trivial interactions: by writing $\hat{S} = \mathbb{1} + i\hat{T}$,

$$\hat{S}^\dagger \hat{S} = (\mathbb{1} - i\hat{T}^\dagger)(\mathbb{1} + i\hat{T}), \tag{25}$$

which gives

$$\hat{T}^\dagger \hat{T} = i(\hat{T}^\dagger - \hat{T}) \neq \mathbb{1}. \tag{26}$$

Putting (26) between the states gives

$$\langle f'|i'\rangle = \langle f|\hat{T}^\dagger \hat{T}|i\rangle = \langle f|i(\hat{T}^\dagger - \hat{T})|i\rangle = i(\langle i|T|f\rangle)^* - i\langle f|T|i\rangle = i(2\pi)^4\delta^{(4)}(p_f - p_i)(A^*_{f\to i} - A_{i\to f}), \tag{27}$$

where we related $\hat{S}$ to the reduced scattering matrix A, same way as in (14). We can write this in terms of imaginary part of the scattering amplitude as follows

$$2iIm(A) = A - A^* = -\langle f|\hat{T}\hat{T}^\dagger|i\rangle = \sum_n \left(\prod_{k=1}^n \int \frac{d^4 q_k}{(2\pi)^4}\frac{1}{2E_k}\right)\langle p_f|T^\dagger|q_k\rangle\langle q_k|T|p_i\rangle, \tag{28}$$

where we inserted complete basis of some intermediate states that we are summing over $|q_k\rangle = |q_1, ...q_n\rangle$. Then (26) can be expressed as

$$A_{i\to f} - A^*_{i\to f} = \sum_n \left(\prod_{k=1}^n \int \frac{d^4 q_k}{(2\pi)^4}\frac{1}{2E_k}\right)(2\pi)^4\delta^{(4)}(\sum_i p_i - \sum_k q_k)A^*_{p_i\to q_k}A_{p_i\to q_k}. \tag{29}$$

Now q_k states can be seen as final states of the amplitudes from initial and final states $|i\rangle$ and $|f\rangle$.

Using the notation

$$d\Pi_\alpha \equiv \prod_k^n \frac{d^4 q_k}{(2\pi)^4 2E_k}. \tag{30}$$

Putting everything together, one gets an Optical theorem:

$$2Im(A_{i\to f}) = \sum_\alpha \int d\Pi_\alpha A^*_{f\to\alpha} A_{i\to\alpha}. \tag{31}$$

For elastic scattering with forward limit case, initial and final momenta are the same so $|i\rangle = |f\rangle$, thus the Optical theorem (31) takes form:

$$2Im(A_{i\to i}) = \sum_\alpha \int d\Pi_\alpha |A_{i\to f}|^2. \tag{32}$$

This expression can be written in terms of conserved energy in the center of mass (CoM) frame E_{CoM} and the cross section $\sigma_{i\to\alpha}$:

$$Im(A_{i\to i}) = 2E_{CoM} p_i \sum_\alpha \sigma_{i\to\alpha}, \tag{33}$$

meaning the imaginary part of the forward scattering amplitude is given by a total cross section of initial states scattering into all possible final states α. We can deduce from this expression a positivity bound due to $E_{CoM} > 0$ and $\sigma_{i\to\alpha} > 0$ by definition, therefore

$$Im(A_{i\to i}) > 0. \tag{34}$$

This is an important result that helps deriving positivity constraints. It also means that for forward scattering limit, $A(s,t,u)$ seen as just a function of s: $A(s)$, in the complex s- plane has only values in the upper half plane. We can analytically extend the amplitude to the lower half plane, by expressing it in terms of the amplitude in the upper half plane. This can be achieved via the Schwartz reflection principle [30]: for a function $A(s)$ analytic in some region, and real when z is real, $A(s)^* = A(s^*)$. Thus, the analytically extended amplitude in the lower half plane equals to the amplitude in the upper half plane but with the complex conjugate of its argument.

1.5 Causality and Analyticity

The condition of causality means that any relevant event occurring in the system may influence the evolution of the system only in the future and not in the past. In a relativistic theory it means that local operators must vanish outside the lightcone: $[\mathcal{O}_1(x), \mathcal{O}_2(0)] = 0,$ for $x^2 > 0$, where x is a 4- vector. Many papers explore how analyticity arises from the causality (see, e.g. [10]). The procedure of how the causality requirement leads to the dispersion relation was first described in [31]. We need the scattering amplitudes to be analytic in the complex s- plane (up to the branch cuts and poles). This, in turn, will lead to the positivity constraint on c.
To understand the analytical structure of the amplitude, one has to understand whether it has any singularities and determine them. To find the poles and branch cuts of this function we use Källén- Lehmann spectral representation, following the method from [32].

Start with a two- point function, the corresponding correlator is $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$. Insert the identity operator expressed as a sum over complete set of states in the Hilbert space:

$$\hat{\mathbb{1}} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3 2E_p(\lambda)} |\lambda_p\rangle\langle\lambda_p|, \tag{35}$$

where λ_p are boosted states of λ_0, which in turn are the eigenstates of Hamiltonian H with momentum 0, so that the boosted states are also eigenstates of H with momentum p. We also assume that the states λ_p are relativistically normalised. Here, $E_p(\lambda) \equiv \sqrt{|p|^2 + m_\lambda^2}$, and m_λ corresponds to the mass of the boosted state. Noting that one- point functions $\langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle$ are just constants that vanish by shift symmetry. The two- point function then can be written as

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3 2E_p(\lambda)} \langle\Omega|\phi(x)|\lambda_p\rangle\langle\lambda_p|\phi(y)|\Omega\rangle. \tag{36}$$

Using Poincare (Lorentz and translation) invariance of the wavefunctions, one can rewrite (36) in terms of the scalar Wightman function $D(x-y; m_\lambda) = \int \frac{d^3p}{(2\pi)^3 2E_p(\lambda)} e^{-ip_\mu(x^\mu - y^\mu)}$. However, taking it one step further and time ordering the correlator ($x^0 > y^0$), yields the Feynman propogator $D_F(x-y; m_\lambda) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_\lambda^2 + i\epsilon} e^{-ip_\mu(x^\mu - y^\mu)}$ with $p^\mu = (E_p(\lambda), \vec{p})$ so that:

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \left|\langle\Omega|\phi(0)|\lambda_0\rangle\right|^2 D_F(x-y; m_\lambda). \tag{37}$$

Defining the spectral density function $\rho(M^2) = \sum_\lambda (2\pi)\delta(M^2 - m_\lambda^2)\left|\langle\Omega|\phi(0)|\lambda_0\rangle\right|^2$ the expression for two- point function in (37) can be rewritten in terms of Källén- Lehmann spectral representation:

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)D_F(x-y;M). \tag{38}$$

By Fourier transforming the two - point function we get the following spectral decomposition:

$$\int d^4x e^{ip\cdot x}\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle = \frac{iZ}{p^2-m^2+i\epsilon} + \int_{4m^2}^\infty \frac{dM^2}{2\pi}\rho(M^2)\frac{i}{p^2-M^2+i\epsilon}, \tag{39}$$

where Z is called field- strength renormalisation, and m is the physical mass of a single particle, defined as energy eigenvalue at rest, to see the schematic form of this spectral function see figure 3.

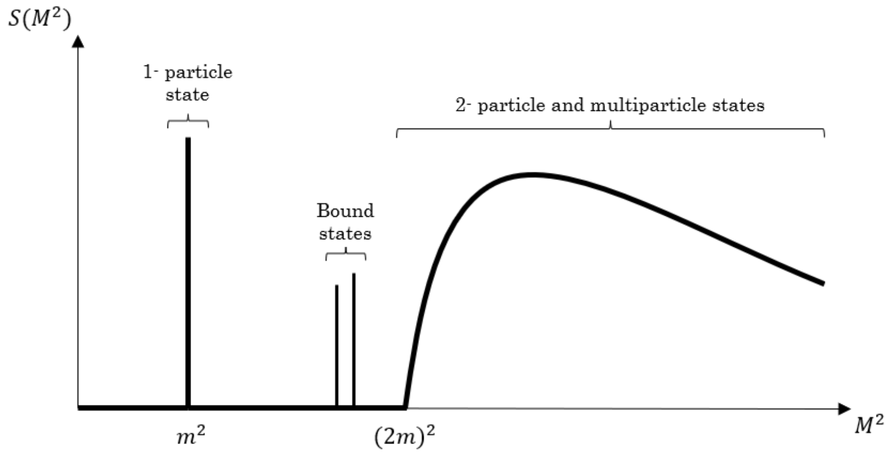

Figure 3: Källén- Lehmann spectral representation $\rho(M^2)$ as a function of M^2, giving a delta function spike at 1- particle state at m^2, some intermediate bound states, and two- particle and multiparticle states starting from $(2m)^2$.

By looking at the analytic structure of this function on complex p^2- plane, the first term corresponds to an isolated simple pole an $p^2 = m^2$ coming from one- particle intermediate states. The second term gives the branch cut beginning at $p^2 = (2m)^2$ along the positive real p^2 axis to infinity, and corresponds to the 2- particle and multiparticle states with rest masses $M > 2m$.

The branch cut correspond to the region in complex s plane where $A(s, 0)$ is not analytic, i.e. for $s \geq 4m^2$ along real axis of s to infinity. The above analytic structure of the two- point function also extends to the scattering amplitude with the addition of a branch cut due to crossing symmetry. In the forward scattering limit $t = 0$, in addition to the pole at $s = m^2$, there is another pole at $u = m^2 \implies s = 4m^2 - u = 3m^2$ chosen to be on the real axis. Although the poles and branch cuts are not analytic, the $A(s)$ function is still analytic everywhere else, including in the region between the poles, $0 < s < 4m^2$, which is often called an unphysical region, due to the fact that there is a Dirac spike at $s = m^2$ corresponding to one- particle obeying on- shell condition, and two- particle state for $s \geq 4m^2$, and nothing else in between (we do not consider contribution from extra spikes of bound states from composite particles), see figure 4. However, these poles can be subtracted from the amplitude.

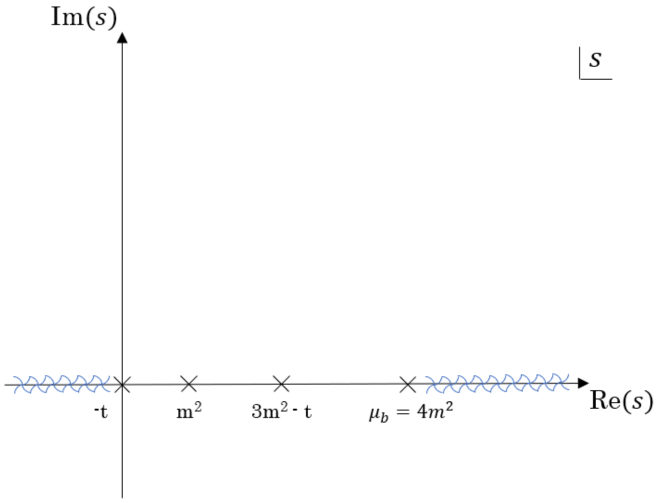

Figure 4: Complex s- plane illustrating singularities on the real axis coming from analytical arguments of amplitude A(s,t =0,u) in forward scattering limit t= 0. The first pole is at m^2, second pole is at $3m^2 - t = 3m^2$, in the t= 0 limit. Here the branch cut on the right side is chosen to start from $\mu_b = 4m^2$ going to $+\infty$, and branch cut on the left side starts at point $-\mu_b + 4m^2 - t = -t = 0$ going to $-\infty$ [33].

We start from the already pole- subtracted amplitude and use Cauchy's integral formula for analytic regions of the amplitude [34]:

$$A(s,t) = \frac{1}{2\pi i} \oint_C d\mu \frac{A(\mu,t)}{\mu - s}, \tag{40}$$

with pole at $\mu = s$, and the closed counterclockwise contour C contains this pole. For A(s,t) being analytic, one can differentiate it using result of Cauchy's differentiation formula

$$\frac{\partial^n A(s,t)}{\partial s^n} = \frac{n!}{2\pi i} \oint_C d\mu \frac{A(\mu,t)}{(\mu - s)^{n+1}}. \tag{41}$$

One can consider any number of differentiation n, but for reasons to be seen later, we choose $n = 2$. For the forward scattering limit $t = 0$, so then u can be expressed as a function of s: $u = 4m^2 - s \implies A(s,t,u) = A(s,0)$ and we compute

$$\frac{\partial^2 A(s,0)}{\partial s^2} = \frac{2!}{2\pi i} \oint_C d\mu \frac{A(\mu,0)}{(\mu - s)^3}. \tag{42}$$

We split the contour integral into a sum of ordinary integrals

$$\frac{1}{2}\frac{\partial^2 A(s,0)}{\partial s^2} = \frac{1}{2\pi i}\Bigg(\int_{-\infty}^{0} d\mu \frac{A(\mu+i\epsilon,0)}{(\mu+i\epsilon-s)^3} + \int_{0}^{-\infty} d\mu \frac{A(\mu-i\epsilon,0)}{(\mu-i\epsilon-s)^3} + \int_{\infty}^{4m^2} d\mu \frac{A(\mu-i\epsilon,0)}{(\mu-i\epsilon-s)^3} + \int_{4m^2}^{\infty} d\mu \frac{A(\mu+i\epsilon,0)}{(\mu+i\epsilon-s)^3} + \int_{C^{\pm\infty}} d\mu \frac{A(\mu,0)}{(\mu-s)^3}\Bigg), \tag{43}$$

where the infinitessimal parameter ϵ was added to ensure the curve is not on the branch cuts/ poles, but just a bit below or above it. See the integration contour on figure 5.

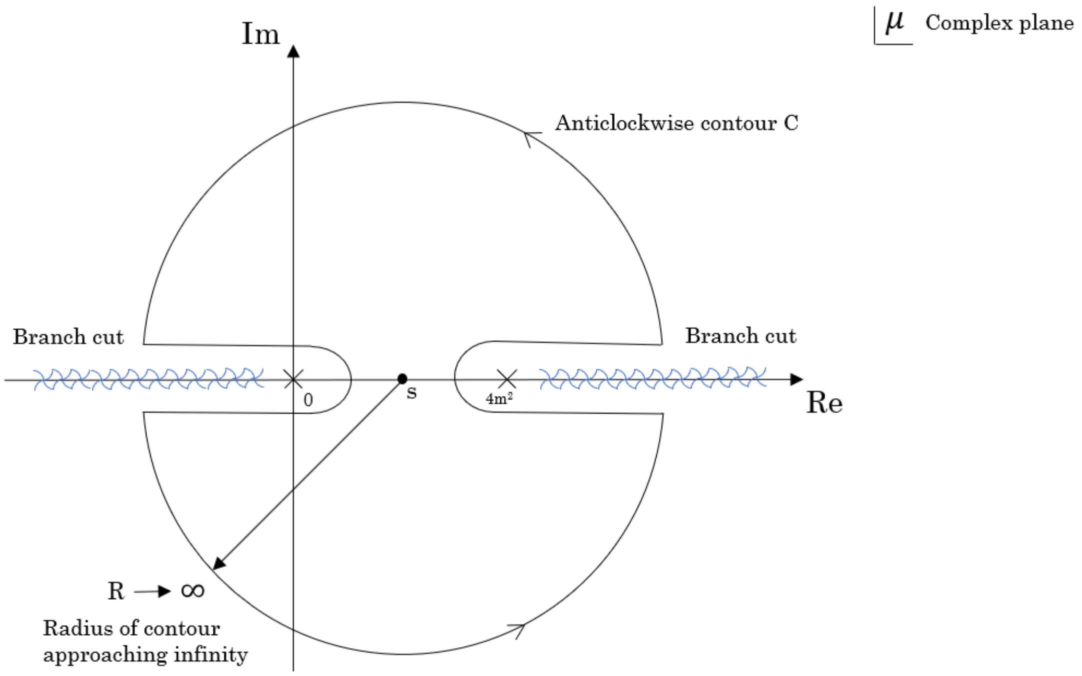

Figure 5: Analytic structure of the 2- 2 scattering amplitude at tree level in the forward scattering limit t=0.

The last term in (43) can be rewritten in terms of polar coordinates, changing $\mu = Re^{i\theta}$ and $d\mu = Rie^{i\theta}d\theta$ for $0 < \theta \leq 2\pi$. At $\pm\infty, \mu - s \approx \mu$, so the integral takes form

$$\frac{1}{2\pi i}\int_{C^{\pm\infty}} d\mu \frac{A(\mu,0)}{\mu^3} = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta A(\mu,0)}{R^2 e^{2i\theta}}. \tag{44}$$

We can now use Froissart- Martin bound [35], which shows how due to analyticity, locality and unitarity of the $\hat{S}$- matrix, the high energy growth of the amplitude is bound guaranteeing that contours in the complex s- plane are closed at infinity [24]. Applying Froissart bound

$\lim_{R\to\infty} A(\mu,0) = \mu ln^2\mu = Re^{i\theta}ln^2(Re^{i\theta})$, (44) becomes

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{Re^{i\theta}ln^2(Re^{i\theta})}{R^2e^{2i\theta}} \sim \frac{ln^2R}{R} \to 0, \text{ as } R\to\infty. \tag{45}$$

The integrand converges as desired, so the last term in (43) vanishes for $R \to \infty$. Note that for $n = 1$, the integrand would be of the form $ln^2(R)$ which diverges for $R \to \infty$. Therefore, the minimal number of differentiation n required for the boundary term to converge is $n = 2$. By 'flipping' the limits of the 4th integral terms in (43) (and picking up the minus sign), the remaining terms can be combined as

$$\frac{1}{2}\frac{\partial^2(A(s,0))}{\partial s^2} = \frac{1}{2\pi i}\Bigg(\int_{-\infty}^{0} d\mu \frac{A(\mu+i\epsilon,0)}{(\mu+i\epsilon-s)^3} - \int_{-\infty}^{0} d\mu \frac{A(\mu-i\epsilon,0)}{(\mu-i\epsilon-s)^3} + \int_{4m^2}^{\infty} d\mu \frac{A(\mu+i\epsilon,0)-A(\mu-i\epsilon,0)}{(\mu-s)^3}\Bigg). \tag{46}$$

Defining the discontinuity function, and rewriting the last term in (46),

$$Disc(A(\mu,0)) = A(\mu+i\epsilon,0) - A(\mu-i\epsilon,0). \tag{47}$$

Meanwhile the first two terms in (46) get a shift in their integration parameter: $\mu \to -\mu, d\mu \to -d\mu$, as well as the integration limits get swapped, resulting in

$$\frac{1}{2}\frac{\partial^2(A(s,0))}{\partial s^2} = \frac{1}{2\pi i}\Bigg(-\int_{0}^{\infty} (-d\mu) \frac{A(-\mu+i\epsilon,0)}{(-\mu-s)^3} + \int_{0}^{\infty} (-d\mu) \frac{A(-\mu-i\epsilon,0)}{(-\mu-s)^3} + \int_{4m^2}^{\infty} d\mu \frac{Disc(A(\mu,0))}{(\mu-s)^3}\Bigg). \tag{48}$$

Shifting focus onto the first two terms in (48): make a substitution $\mu = \mu' - 4m^2$ and $d\mu = d\mu'$. Considering the limits of $\int_0^\infty$: $\lim_{\mu\to 0}(\mu' = 4m^2 - \mu) = 4m^2$ and $\lim_{\mu\to\infty}\mu' = +\infty$. Therefore, limits of the integral become $\int_{4m^2}^\infty$ and first two terms of (48) now become

$$\frac{1}{2\pi i}\Bigg(\int_{4m^2}^{\infty} d\mu' \frac{A(-\mu'+4m^2+i\epsilon,0)}{(-\mu'+4m^2-s)^3} - \int_{4m^2}^{\infty} d\mu' \frac{A(-\mu'+4m^2-i\epsilon,0)}{(-\mu'+4m^2-s)^3}\Bigg) \tag{49}$$

Substituting the on-shell condition $4m^2 - s = u$ into the denominator to get

$$\frac{1}{2\pi i}\Bigg(\int_{4m^2}^{\infty} d\mu' \Bigg(-\frac{A(-\mu'+4m^2+i\epsilon,0)}{(\mu'-u)^3} + \frac{A(-\mu'+4m^2-i\epsilon,0)}{(\mu'-u)^3}\Bigg)\Bigg) \tag{50}$$

Due to crossing symmetry, amplitudes satisfy,

$$A(s,0,u) = A(u,0,s) = A(4m^2-s,0,s) \tag{51}$$

then,

$$A(-\mu' + 4m^2 - i\epsilon, 0) = A(\mu' + i\epsilon, 0). \tag{52}$$

So (50) can be rewritten using (52):

$$\frac{1}{2\pi i}\int_{4m^2}^{\infty} d\mu' \bigg(\frac{A(\mu' + i\epsilon, 0) - A(\mu' - i\epsilon, 0)}{(\mu' - u)^3} \bigg) \tag{53}$$

By changing the integration parameter $\mu' \to \mu$ and rewriting equation (53) with discontinuity function (47), and putting it back into (48) gives:

$$\frac{1}{2}\frac{\partial^2(A(s,0))}{\partial s^2} = \frac{1}{2\pi i}\int_{4m^2}^{\infty} d\mu \bigg(\frac{DiscA(\mu, 0, u)}{(\mu - s)^3} + \frac{DiscA(\mu, 0, u)}{(\mu - u)^3} \bigg). \tag{54}$$

From Schwartz reflection principle $A(s^*) = A(s)^*$ it follows that

$$2iImA(\mu) = A(s) - A(s)^* = A(s) - A(s^*) = A(\mu + i\epsilon, 0) - A(\mu - i\epsilon, 0) = Disc(A(\mu, 0)) \tag{55}$$

Thus, the expression (54) becomes

$$\frac{1}{2}\frac{\partial^2(A(s,0))}{\partial s^2} = \frac{1}{\pi}\int_{4m^2}^{\infty} d\mu Im(A(\mu)) \bigg(\frac{1}{(\mu - s)^3} + \frac{1}{(\mu - u)^3} \bigg) \tag{56}$$

In section 1.4 it is shown how unitarity results in Optical theorem: $Im(A(\mu)) > 0$. Also terms, $\frac{1}{(\mu - s)^3}$ and $\frac{1}{(\mu - u)^3} = \frac{1}{(\mu - 4m^2 + s)^3}$, are positive, as long as s is chosen to be in the interval $0 < s < 4m^2$. This, in turn, leads to the formulation of the positivity constraint,

$$\frac{\partial^2 A(s,0)}{\partial s^2} > 0. \tag{57}$$

This result can be generalised to any number n of derivatives taken (although, we do require $n > 2$ to ensure the contour integrals $C^{\pm\infty}$ are infinite in the limit $\mu \to \infty$ so it vanishes due to Froissart-Martin bound):

$$\bigg[\frac{\partial^n A(s,0)}{\partial s^n}\bigg]_{EFT} = \bigg[\frac{n!}{\pi}\int_{\mu_b}^{\infty} d\mu Im(A(\mu, 0)) \bigg(\frac{1}{(\mu - s)^{n+1}} + \frac{1}{(\mu - u)^{n+1}} \bigg)\bigg]_{UV} > 0, \tag{58}$$

where μ_b is the scale at which the branch cut begins [24].

1.6 Flavour Constraints from Unitarity and Analyticity

The results from unitarity and analyticity of scattering amplitudes has many applications, one of them will be reviewed in this section: the resulting constraints on fermion operators in the SMEFT as described in [36]. This letter demonstrates how regardless of the form of new higher energy physics that might emerge, if it satisfies the axioms of unitarity and analyticity it will produce flavour constraints on corresponding interactions. It was shown previously how by imposing analyticity on the scattering amplitude in the complex s- plane, in the forward scattering limit, by imposing Froissart bound and Optical Theorem, one achieves positivity of the n^{th} derivative w.r.t. s of scattering amplitude (for $n \geq 2$) (58). In the section 2 it is shown how this positivity constraint results in positivity bounds of the Wilson coefficients. This principle of applying IR consistency to find bounds in the corresponding EFT is also used to constrain fermionic scatterings.

As mentioned in equation (19), in SMEFT the Lagrangian may include higher mass dimension operators. For the fermion section of SMEFT dimension- 8 operators are considered, however there are also works exploring analyticity effect on fermion scattering via mass dimension- 6 operators as in [37]. It should be noted that for higher- dimension operators the UV scale is much greater than masses of fermions in the SM. Here, the operators are taking form of $c_{mnpq}\partial^2(\bar{\psi_m}\psi_n)(\bar{\psi_p}\psi_q)$, with flavour indices m,n,p,q of the fermionic fields and the corresponding coupling (Wilson coefficient) c, additionally it should be noted that one requires an even number of each flavour. The fields considered in this paper ([36]), are

- Left- handed quark Q and lepton L multiplets, where Q and L are doublets of SU(2)
- Right- handed up and down quarks u and d, and lepton e, where u,d as well as Q are triplets of SU(3).

Each field has a generation (or family) index running from 1 to N_f, in Standard Model $N_f = 3$. Working in the unbroken phase of the SMEFT, fermions are considered effectively massless $\not{\partial}\psi = 0$. To ensure Lagrangian operators are written in terms of the corresponding generators, for fields ψ_m charged currents are defined in terms of the corresponding symmetry group generators: τ^I for SU(2) and T^a for SU(3) as:

$$\begin{aligned} J^\mu[\psi]_{mn} &= \bar{\psi}_m\gamma_\mu\psi_n, \quad J^\mu[\psi]^I_{mn} = \bar{\psi}_m\tau^I\gamma_\mu\psi_n, \\ J^\mu[\psi]^a_{mn} &= \bar{\psi}_m T^a\gamma_\mu\psi_n, \quad J^\mu[\psi]^I_{mn}a = \bar{\psi}_m\tau^I T^a\gamma_\mu\psi_n. \end{aligned} \tag{59}$$

The corresponding (self- quartic) operators are then written as

$$
\begin{aligned}
\text{for } \psi = Q, L, e, u, d : \mathcal{O}_1[\psi] &= -c^{\psi,1}_{mnpq}\Big(\partial_\mu J_\nu[\psi]_{mn}\Big)\Big(\partial^\mu J^\nu[\psi]_{pq}\Big) \\
\text{for } \psi = Q, L : \mathcal{O}_2[\psi] &= -c^{\psi,2}_{mnpq}\Big(\partial_\mu J_\nu[\psi]^I_{mn}\Big)\Big(\partial^\mu J^\nu[\psi]^I_{pq}\Big) \\
\text{for } \psi = Q, u, d : \mathcal{O}_3[\psi] &= -c^{\psi,3}_{mnpq}\Big(\partial_\mu J_\nu[\psi]^a_{mn}\Big)\Big(\partial^\mu J^\nu[\psi]^a_{pq}\Big) \\
\text{for } \psi = Q : \mathcal{O}_4[Q] &= -c^{Q}_{mnpq}\Big(\partial_\mu J_\nu[\psi]^{Ia}_{mn}\Big)\Big(\partial^\mu J^\nu[\psi]^{Ia}_{pq}\Big),
\end{aligned}
\tag{60}
$$

where c_{mnpq} is a Wilson coefficient written as a tensor in flavour space, which due to its self-hermition and symmetry conditions leaves $\frac{N_f^2(N_f^2+1)}{2}$ real operators for each field choice in each line. There are also other operators mixing different fields (cross- quartic), resulting in the total basis of 10 sets of operators. One of the examples discussed in the paper is 2-2 scattering of right-handed leptons e, with corresponding operator $\mathcal{O}_1(e)$, then the s^2 contribution to the forward amplitudes is shown to be:

$$
A_{e^-e^+e^-e^+} = A_{e^-e^-e^+e^+} = 4c^{e,1}_{mnpq}\alpha_m\beta_n\beta^*_p\alpha^*_q s^2, \tag{61}
$$

where α, β are vectors corresponding to the incoming particle states. Utilising result coming from the unitarity and analyticity arguments expressed in (57) and (58), one obtains the following positivity bound: $c^{e,1}_{mnpq}\alpha_m\beta_n\beta^*_p\alpha^*_q > 0$. Expressing this in terms of density matrices on Hilbert space of dimension N_f: $\rho^\alpha_{mq} = \alpha_m\alpha^*_q$, and given that the density matrices ρ^α_{mq} and ρ^β_{np} are pure (unit trace), implies condition on the Wilsonian coefficient: $c^{e,1}_{\alpha\beta} > 0$. This result was then implemented as flavour conserving operators must have positive coeffic, ients. with the implication that interaction violating lepton lumber is allowed provided existence of the flavour- conserving operators. Thus, there is relation between violation and conservation of flavour quantum numbers which can, in turn, be related to CP- violation condition described in [38]. This relation was then derived for each scattering, corresponding to self- quartic and cross- quartic operators, all resulting in positive bounds on corresponding Wilsonian coefficients. To summarise, the flavour-violating Wilsonian coefficients are bounded by the flavour- conserving coefficients. These results were investigated in the lepton flavour- violating $\mu \to e^+e^-e^+$ decay proposed experiment Mu3e [39].

2 Scattering of Scalar Fields

In this section we consider 2-2 scattering, with corresponding amplitude A(s,t) expressed in terms of 2 Mandelstam invariants $s = (p_1 + p_2)^2, t = (p_1 - p_3)^2$. As discussed before, at high energies we assume the tenets of the S- program: it should be analytic, unitary, local and obey the crossing symmetry. These S- matrix axioms at high energies (UV- complete theory) will lead to constraints on Wilson coefficients for low energy EFT. Consider the elastic 2-2 scattering $\phi(x), \phi(x) \to \phi(x), \phi(x)$ with a Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{c}{4!\Lambda^4} \partial^\mu \phi \partial_\mu \phi \partial^\nu \phi \partial_\nu \phi, \tag{62}$$

where Λ is the energy cut- off. Although no inital constraint is applied on c from a low energy EFT part of view: it can be both positive or negative, later it will be shown that c can only be positive for theory to be UV- completed. The scattering matrix we wish to compute is

$$S = \lim_{T \to \infty} \langle p_3, p_4|_T \hat{S} |p_1, p_2\rangle_{-T}, \tag{63}$$

which can be computed by performing a reduction procedure to get the LSZ formula [32], which further can be Fourier transformed and after performing spatial and momentum integral, imposing on- shell condition one gets an expression

$$\begin{aligned} {}_{out}\langle q_1, q_2 | p_1, p_2 \rangle_{in} \Big|_{p_{1,2}^2, q_{1,2}^2 \to m^2} &= trivial + \prod_{j=1,2} (\frac{q_j^2 - m^2}{i}) \prod_{i=1,2} (\frac{p_i^2 - m^2}{i}) \\ &\times \langle \Omega | T \hat{\phi}(q_1) \hat{\phi}(q_2) \hat{\phi}(-p_1) \hat{\phi}(-p_2) | \Omega \rangle, \end{aligned} \tag{64}$$

where the product terms will exactly cancel out factors from Feynman propagators, leaving us with "amputated" propagation terms.

To compute the correlator one needs the interacting Hamiltonian

$$\mathcal{H} = \int d^3x (\pi^2 - \mathcal{L}) = H_0 + H_{int}. \tag{65}$$

Here the Hamiltonian has a complex structure and for simplicity we will use interacting part of the Lagrangian without loss of any information:

$$e^{-i \int H_{int} dt} = e^{i \int d^4x \mathcal{L}_{int}} = e^{\frac{ic}{4!\Lambda^4} \int d^4x \partial^\mu \phi \partial_\mu \phi \partial^\nu \phi \partial_\nu \phi} \tag{66}$$

Computing the correlation function

$$\begin{aligned}\langle\Omega|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle &= \frac{\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)e^{i\int d^4x\mathcal{L}_{int}}|0\rangle}{\langle 0|Te^{i\int d^4x\mathcal{L}_{int}}|0\rangle} \\ &= \langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)e^{\frac{ic}{4!\Lambda^4}\int d^4x\partial^\mu\phi\partial_\mu\phi\partial^\nu\phi\partial_\nu\phi}|0\rangle,\end{aligned} \tag{67}$$

where the denominator's main role is to cancel out all the contribution coming from bubble (disconnected) Feynman diagrams. Recalling the field $\hat{\phi}$ can be written in terms of creation and annihilation ladder operators:

$$\hat{\phi}(x) = \int \frac{d^4p}{(2\pi)^4}(\hat{a}(p)e^{-ip\cdot x} + \hat{a}^\dagger(p)e^{ip\cdot x}), \tag{68}$$

where the annihilation mode $\int \frac{d^4p}{(2\pi)^4}\hat{a}(p)e^{-ip\cdot x}$ corresponds to incoming particles at the vertex, and the creation mode $\int \frac{d^4p}{(2\pi)^4}\hat{a}^\dagger(p)e^{ip\cdot x}$ corresponds to the outgoing particles. So the derivative of the field for incoming particles gives

$$\partial_\mu\phi = \partial_\mu(\hat{a}(p)e^{-ip\cdot x}) = -ip_\mu\hat{a}(p)e^{-ip\cdot x} = -ip_\mu\phi, \tag{69}$$

and similarly for outgoing particles

$$\partial_\mu\phi = \partial_\mu(\hat{a}^\dagger e^{ip\cdot x}) = ip_\mu\phi. \tag{70}$$

The main contribution to the scattering part is the tree level term, so expanding exponent in (67) the corresponding term gives

$$\begin{aligned}\frac{ic}{4!\Lambda^4}\int d^4x\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\Big[&(-ip_1^\mu\phi(x))(-ip_{2\mu}\phi(x))(ip_3^\nu\phi(x))(ip_{4\nu}\phi(x)) \\ &+ (-ip_1^\mu\phi(x))(-ip_2^\nu\phi(x))(ip_{3\mu}\phi(x))(ip_{4\nu}\phi(x)) \\ &+ (-ip_1^\mu\phi(x))(-ip_2^\nu\phi(x))(ip_{3\nu}\phi(x))(ip_{4\mu}\phi(x))\Big]|0\rangle,\end{aligned} \tag{71}$$

where we wrote 3 different ways of contracting momenta, corresponding to s-, t-, u- channels. The expression in (71) can be computed via Wick's contraction, where incoming fields $\phi(x_i)$, (with i=1,...,4) are contracted with fields from the vertex $\phi(x)$. Counting the permutations of such contractions gives an extra factor of 24 which cancels out the 4! factor in front of the expression, and 3 channels give an extra factor of $\frac{1}{3}$. Each Wick contraction will give a Feynman propagator factor

$$\frac{ic}{4!\Lambda^4}\int d^4x(-ip_1^\mu)(-ip_{2\mu})(-ip_3^\nu)(-ip_{4\nu})\frac{1}{(2\pi)^{12}}\int d^4k_1\int d^4k_2\int d^4k_3\frac{ie^{-ik_1\cdot(x_1-x)}}{k_1^2-m^2+i\epsilon}\times\dots$$
$$\times\frac{ie^{-ik_4\cdot(x_4-x)}}{k_4^2-m^2+i\epsilon}+perm(p_{2\leftrightarrow3}\text{ and }p_{2\leftrightarrow4}), \tag{72}$$

where $k_4=-(k_1+k_2+k_3)$, therefore there is no integral for it. Fourier transforming at each external point $\int d^4x_ie^{ip_i\cdot x_i}$ gives an expression for $\langle\Omega|T\phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4)|\Omega\rangle$. That allows us to perform the vertex and external point integrals, and then one gets delta function factors $(2\pi)^4\delta^{(4)}(p_i-k_i)$ so then momenta integrals $\int\frac{d^4k_i}{(2\pi)^4}$ can be performed with delta functions setting $k_i=p_i$. Since there is no momentum integral for k_4, there will be one delta function still left in the expression: $(2\pi)^4\delta^{(4)}(p_4-k_4)=(2\pi)^4\delta^{(4)}(p_4+p_1+p_2+p_3)$. This can now be plugged back into (64).
As mentioned before due to expression in (64) the factors of $\frac{i}{k_i^2-m^2+i\epsilon}$ are canceled out (taking $\epsilon\to0$) so (64) now can be written as simply:

$$\begin{aligned}{}_{out}\langle q_1,q_2|p_1,p_2\rangle_{in}\Big|_{p_{1,2}^2,q_{1,2}^2\to m^2}&=\frac{ic}{3\Lambda^4}(2\pi)^4\delta^{(4)}(q_1+q_2-p_1-p_2)\\&\times\Big((p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)\Big).\end{aligned} \tag{73}$$

Comparing with expression in (14), allows to write

$$\begin{aligned}A^s_{p\to q}&=\frac{c}{3\Lambda^4}(p_1\cdot p_2)(p_3\cdot p_4)\\A^t_{p\to q}&=\frac{c}{3\Lambda^4}(p_1\cdot p_3)(p_2\cdot p_4)\\A^u_{p\to q}&=\frac{c}{3\Lambda^4}(p_1\cdot p_4)(p_2\cdot p_3).\end{aligned} \tag{74}$$

For the rest mass of the particle being so much smaller than its energy one can apply the Eikonal approximation described in (2): $s=(p_1+p_2)^2\approx2p_1\cdot p_2$, similarly for u- and t- channels, giving an extra $\frac{1}{2^2}$ factor. The expression in (74) (dropping $p\to q$ subscript) becomes

$$\begin{aligned}A^s&=\frac{c}{12\Lambda^4}s^2\\A^t&=\frac{c}{12\Lambda^4}t^2\\A^u&=\frac{c}{12\Lambda^4}u^2\end{aligned} \tag{75}$$

In the forward scattering limit $t=0$, the on-shell condition: $s+u+t=4m^2$, gives $u^2=(4m^2-s)^2$ so the full amplitude is

$$A=A^s+A^u=\frac{c}{12\Lambda^4}(s^2+u^2)=\frac{c}{12\Lambda^4}(16m^4-8m^2s+2s^2), \tag{76}$$

From the analytic arguments of the amplitude described is section 1.5, where we showed how Cauchy's integral formula together with Optical theorem results in positivity bound, it follows that:

$$\frac{\partial^2 A(s_0, t=0)}{\partial s_0^2} = \frac{c}{3\Lambda^4} > 0, \tag{77}$$

giving the important result that $c > 0$. Thus, the initially unconstrained coefficient c now is bound to be positive. This is the positivity bound [40], which in [11] is shown to be consitent with causality assumption based on speed of propagation.

2.1 Integration of heavy fields at high energy scales

Including fields heavier than the cutoff scale Λ in the theory brings us to the UV region, where the scattering might include some intermediate process, see a sketch 6.

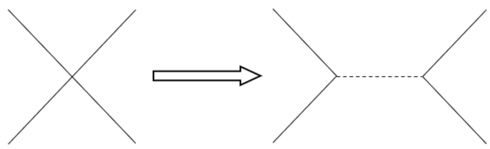

Figure 6: A schematic illustration of Feynman diagram for 2-2 scattering process at tree level in IR and UV regions.

Recall for initial IR Lagrangian the action is:

$$S_{EFT} = \int d^4x(-\frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2\phi(x)^2 + \frac{c}{4!\Lambda^4}\partial^\mu\phi\partial_\mu\phi\partial^\nu\phi\partial_\nu\phi), \tag{78}$$

for a single scalar field with light mass m, $m \ll \Lambda$, allowing to use the eikonal approximation for computing amplitudes of 2-2 elastic forward scattering. As result, one obtains a constraint on c, $c > 0$, arising from analyticity and unitarity of the S-matrix.

Now we look at the massive case and how the same constraint result on c can be obtained from an explicit UV completion. This can be done by completing the initial scalar theory within the UV regime, i.e. high energy, $M \gg \Lambda$.

By adding a massive field $\chi(x)$ with mass way above energy scale $M \gg \Lambda$, the complete theory becomes as follows,

$$S_{UV} = \int d^4x(-\frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}\partial^\nu\chi(x)\partial_\nu\chi(x) - \frac{1}{2}m^2\phi(x)^2 - \frac{1}{2}M^2\chi(x)^2 + \frac{\alpha}{\Lambda}\partial^\mu\phi(x)\partial_\mu\phi(x)\chi(x)). \tag{79}$$

Or, alternatively, the Lagrangian is

$$\mathcal{L}_{UV} = -\frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}\partial^\nu\chi(x)\partial_\nu\chi(x) - \frac{1}{2}m^2\phi(x)^2 - \frac{1}{2}M^2\chi(x)^2 + \frac{\alpha}{\Lambda}\partial^\mu\phi(x)\partial_\mu\phi(x)\chi(x), \quad (80)$$

where α is the coupling factor.

We begin by 'integrating out' the massive field $\chi(x)$, which at tree level means the following process:

1. Use the Euler- Lagrange equation to obtain the equation of motion for $\chi(x)$.
2. Solve the subsequent equation of motion.
3. Substitute the solution back into the UV Lagrangian one started with to obtain the expression for low energy (EFT) Lagrangian.

To begin, the Euler- Lagrange equation gives,

$$-M^2\chi + \frac{\alpha}{\Lambda}\partial^\mu\phi\partial_\mu\phi - \partial_\mu(-\partial^\mu\chi) = 0 \quad (81)$$

which can be rewritten as

$$\frac{\alpha}{\Lambda}\partial^\mu\phi\partial_\mu\phi = M^2\chi - \partial_\mu\partial^\mu\chi. \quad (82)$$

To solve for χ one takes the plane wave form solution

$$\chi(x) = \exp(ik_\mu\chi^\mu), \quad (83)$$

and the derivative

$$\partial^\mu\chi(x) = ik^\mu\chi(x). \quad (84)$$

(84) can be substituted into (82), and the following can be obtained,

$$\frac{\alpha}{\Lambda}\partial^\mu\phi\partial_\mu\phi = M^2\chi - (ik^\mu)^2\chi = (M^2 + k^2)\chi. \quad (85)$$

Or alternatively, χ can be expressed in terms of the operator $\Box$:

$$\chi = \frac{\alpha}{\Lambda}\frac{1}{M^2 - \Box}(\partial^\mu\phi\partial_\mu\phi). \quad (86)$$

Taking expression in (85) and multiplying by χ, gives

$$-\chi\Box\chi = \frac{\alpha}{\Lambda}\chi\partial^\mu\phi\partial_\mu\phi - M^2\chi^2. \quad (87)$$

Integrating both sides,

$$-\int d^4x\chi\Box\chi = \int d^4x(\frac{\alpha}{\Lambda}\chi\partial^\mu\phi\partial_\mu\phi - M^2\chi^2). \qquad (88)$$

Take left hand side (LHS) of (88) and integrate by parts to get

$$\int d^4x\chi\Box\chi = -\int d^4x\partial_\mu\chi\partial^\mu\chi, \qquad (89)$$

where the boundary term vanishes for $x \to \infty$. By substituting this result from (89) into (88),

$$\partial_\mu\chi\partial^\mu\chi = \left(\frac{\alpha^2}{\Lambda^2(M^2-\Box)}\partial^\mu\phi\partial_\mu\phi\right)\partial^\nu\phi\partial_\nu\phi - M^2\chi. \qquad (90)$$

Using (90) to substitute back into the UV regime Lagrangian (80), one gets,

$$\mathcal{L}_{UV} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{\alpha^2}{2\Lambda^2}\left(\frac{1}{M^2-\Box}\partial^\mu\phi\partial_\mu\phi\right)\partial^\nu\phi\partial_\nu\phi. \qquad (91)$$

To simplify the last term in (91), use series expansion $\frac{1}{1-x} \approx 1 + x + x^2 + \cdots$, leaving only the leading term, getting

$$\mathcal{L}_{UV} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{\alpha^2}{2\Lambda^2}\frac{1}{M^2}\partial^\mu\phi\partial_\mu\phi\partial^\nu\phi\partial_\nu\phi. \qquad (92)$$

It is sufficient now to equate (82) and (92). Which as a result produces the following expression of c

$$c = \frac{12\Lambda^2\alpha^2}{M^2}, \qquad (93)$$

which is automatically positive, given that Λ, α, M are real. Thus, at tree level one obtains the same constraint on Wilson coefficient c via both top- bottom and bottom- up methods. This shows that the assumptions made in deriving positivity bounds are consistent.

3 Scattering of Vector Fields

For a low energy EFT it was shown that scattering amplitudes must satisfy certain inequalities, so the theory can be UV- completed and agree with local, unitary, analytic and Lorentz-invariant requirements. Unlike for the scalar case with zero spin particles, for particles with nonzero spin the extension of those results is more subtle due to non- trivial crossing relations.

3.1 Massless Case: Maxwell field

First I will write brief recap of massless vector field. One can write Maxwell (vector) field:

$$\hat{A}^{\mu}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2E(\vec{k})} \sum_{\alpha=0}^{3} \epsilon^{\mu}_{\alpha}(\vec{k}) \Big(\hat{a}_{\alpha}(\vec{k}) e^{(-ik\cdot x)} + \hat{a}^{\dagger}_{\alpha}(\vec{k}) e^{(ik\cdot x)} \Big), \tag{94}$$

where $\hat{A}_{\mu}$ obeys the Heisenberg picture equal time commutation relation and the ladder operators $\hat{a}_{\alpha}$ and $\hat{a}^{\dagger}_{\alpha}$ obey the usual commutation relations of form $[a, a^{\dagger}]$.

The polarization vector is $\epsilon^{\mu}_{s}(\vec{k}) = (0, \vec{e_s}(\vec{k}))$, and one can choose the basis

$$\epsilon^{\mu}_{0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{\mu}_{1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{\mu}_{2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon^{\mu}_{3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{95}$$

so that together with the on- shell wavevector, the polarization vectors satify: $k\cdot\epsilon_{(1,2)}(\vec{k}) = 0, \quad k\cdot\epsilon_{(0)}(\vec{k}) = |\vec{k}|, \quad k\cdot\epsilon_{(3)}(\vec{k}) = -|\vec{k}|$. The polarizations for $\alpha = 0$ are called scalar polarizations, for $\alpha = 1, 2$ are the physical (or transverse) polarizations, and $\alpha = 3$ are the longitudinal polarizations. One should note that in a scattering polarization is not an invariant quantity: there is no conservation of polarisation in a scattering. Unless the experiment is set up in such a way that we have definite polarization, when a polarizer is applied to the incoming and outgoing particles so that all the polarization is known. The propagation of the Maxwell field is given by the correlator:

$$\langle 0|T\hat{A}_{\mu}(x)\hat{A}_{\nu}(x)|0\rangle = -\eta_{\mu\nu} D_F(x-y), \tag{96}$$

where D_F is the massless scalar Feynman propagator,

$$D_F = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik\cdot x}}{k^2 + i\epsilon}. \tag{97}$$

Decomposing the Maxwell field into the annihilation and creation modes $\hat{A}_\mu = A^+_\mu + A^-_\mu$, taking its derivative one gets for ingoing particles:

$$\partial_\mu (A^+)^\mu = -i \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} e^{-ik\cdot x} \sum_{\alpha=0}^{3} k \cdot \epsilon_\alpha(\vec{k}) \hat{a}_\alpha(\vec{k}), \tag{98}$$

and outgoing particles:

$$\partial_\mu (A^-)^\mu = i \int \frac{d^3k}{(2\pi)^3 2E(\vec{k})} e^{ik\cdot x} \sum_{\alpha=0}^{3} k \cdot \epsilon_\alpha(\vec{k}) \hat{a}^\dagger_\alpha(\vec{k}). \tag{99}$$

Unlike in the scalar case, when considering vectors, e.g. photons (Maxwell) fields, we need the context of QED and its concepts of spinors, e.g. electrons. From the $\mathcal{L}_{QED}$ one gets,

$$\hat{H}_I = e \int d^3x \bar{\psi} \gamma^\mu \psi A_\mu, \tag{100}$$

which is used to compute the correlator corresponding to a scattering.

3.2 Massive case: Proca field

In this section I will cover some basics of the massive vector field: Proca field, following [41]. Recall that the Klein-Gordon describes the scalar field giving the equation of motion, $(\Box + m^2)\phi = 0$. A Dirac field describes the spinor field which gives equation of motion: $(i\partial\!\!\!/ - m)\psi = 0$. A Maxwell field describes the massless vector field, with Maxwell equation $\Box A^\mu = 0$ in Lorenz gauge, coming from Maxwell Lagrangian:

$$\mathcal{L}_{Maxwell} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A_\nu)^2. \tag{101}$$

The Proca field describes the massive vector field. Then the Lagrangian of the Proca field is:

$$\mathcal{L}_{Proca} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\nu - j_\mu A^\mu \tag{102}$$

where, $j^\mu = (\rho, \vec{j})$ is a current.

Using the Euler- Lagrange equation on the Proca Lagrangian one gets,

$$\Box A^\nu - \partial^\nu (\partial_\mu A^\mu) + m^2 A^\nu = j^\nu = 0, \tag{103}$$

for zero- current. Note for the Maxwell field $m = 0$, this expression simplifies to the Maxwell equation of motion $\Box A^\mu = 0$. By differentiating (103) on both sides with ∂_ν one gets the expression

$$\partial_\nu A^\nu = \frac{1}{m^2}\partial_\nu j^\nu = 0, \tag{104}$$

for conserved current $\partial_\nu j^\nu = 0$ or absence of current $j^\nu = 0$. Putting this back into (103) one gets an equation of motion of the same form as Klein- Gordon:

$$(\Box + m^2)A^\nu = 0. \tag{105}$$

From now on I will follow the approach described in the section 3 in [42] for a massive spin - 1 field and review the steps. The EFT is constructed using the bottom - up method, and then with the assumption of UV- completion constraints and positivity bounds are derived. Starting with a massless vector field A_μ (or Maxwell field), the corresponding EFT Lagrangian will have terms with field strength tensor $F_{\mu\nu}$ and its derivatives, combined with cutoff Λ_A and an undetermined coupling coefficient g_* appearing in the Lagrangian as $\frac{1}{g_*^2}$. Then one should note that for $g_* \sim 1$ the theory would have strong coupling near Λ_A, and one would have to subtract the light loops, see [43] for improved positivity bounds used to deal with loops. However, for $g_* \ll 1$, and if one applies positivity bound on it so it $g_* > 0$, then contribution of these loops will be neglegible and one can still consider only tree- level positivity bounds.
The massive spin- 1 field appears via symmetry breaking process, that generates new terms in the Lagrangian with $\phi_\mu = D_\mu\phi = \partial_\mu\phi + mA_\mu$ (noting that ϕ here transforms non- linearly under gauge symmetry transformation). This process is described via the Stückelberg mechanism: introducing a scalar field which makes an Abelian gauge theory massive but preserves the gauge invariance [44]. The corresponding cutoff scale now is Λ_ϕ and for the low energy EFT we have $m \ll \Lambda_{A/\phi}$.
Considering 2-2 scattering, at the tree- level to compute the amplitude, a unitary gauge is applied $\phi = 0$, as well as $\Lambda_\phi^3 = m\Lambda_A^2$ is taken, then the relevant terms in the Lagrangian are:

$$\begin{aligned} g_*^2\mathcal{L}_{EFT} \supset & -\frac{1}{4}F^\nu_\mu F^\mu_\nu - \frac{1}{2}m^2 A_\mu A^\mu + \frac{m^4 a_0}{\Lambda_\phi^4}(A_\mu A^\mu)^2 \\ & + \frac{m^4}{\Lambda_\phi^6}\Big(a_3 A_\mu A_\nu \partial^\mu A_\rho \partial^\nu A^\rho + a_4 A_\mu A_\nu \partial_\rho A^\mu \partial^\rho A^\nu + a_5 A_\mu A^\mu \partial_\alpha A_\beta \partial^\beta A^\alpha\Big) \\ & + \frac{1}{\Lambda_A^4}\Big(c_1 F^\mu_\nu F^\nu_\rho F^\rho_\sigma F^\sigma_\mu + c_2 (F^2_{\mu\nu})^2\Big) + \frac{m^4}{\Lambda_\phi^6}\Big(C_1 A_\mu A^\nu F^{\alpha\mu}F_{\alpha\nu} + C_2 F^2_{\mu\nu} A_\alpha A^\alpha\Big). \end{aligned} \tag{106}$$

Using the helicity and transversity formalism described in [45]: spin projections are orthogonal to the scattering plane; for 2-2 scattering of massive particles with a complete set of polarizations one can derive positive bounds, both at and away from the forward scattering limit. Then to find the scattering amplitudes, one uses polarization vectors ϵ^μ in the transversity basis (107):

$$\begin{aligned} \epsilon^\mu_{\tau=\pm 1} &= \frac{i}{\sqrt{2}m}(p, E\sin(\theta) \pm im\cos(\theta), 0, E\cos(\theta) \mp im\sin(\theta)) \\ \epsilon^\mu_{\tau=0} &= (0,0,1,0) \end{aligned} . \tag{107}$$

Then by taking each term in the Lagrangian (106) and computing the respective elastic scattering amplitudes at tree- level in the transversity basis up to a factor of $\frac{1}{g_*^2}$, in the forward scattering limit $t = 0$, following procedure in [46], one gets a set of expressions for the amplitudes. Invoking result in (58), one sums all the obtained amplitudes A_i, just like it was explicitly done for the scalar field 2-2 scattering, to use the positivity bound

$$\sum_i \frac{\partial^2 A_i}{\partial s^2} > 0. \tag{108}$$

one gets the following set of requirements for positivity bounds:

$$a_0 > 0,\ c_1 > 0,\ c_1 + 2c_2 > 0,\ a_3 + C_1 > 0,\ \text{for } \Lambda_A^2 = \Lambda_\phi^3/m. \tag{109}$$

The positivity bounds for weak vector boson scattering (VBS) (e.g. $ZZ \to ZZ$ scattering) at the LHC is discussed in [47], where quartic- gauge- boson coupling is described by 18 dimension-8 operators, and by assuming UV- completion of the theory new constraints on coefficients of these operators are derived, leaving only 2% of the initial full parameter space that allows UV-completion. One should also note that only even - dimensional operators conserve baryon and lepton numbers, so usually the focus is on dimension- 6 and dimension- 8 operators [48]. For elastic scattering processes mixing different particle species, positivity bounds on the transversal quartic-gauge-boson couplings are derived in [49], showing that they exclude $\approx$ 99.3% of the parameter space at the LHC.

4 Conclusion

In this project I reviewed some basic concepts of the Effective Field Theory. EFT describes relevant physics below an energy cutoff Λ, what is called an IR region, without including physics at higher energy, UV region, into the framework. Quantities in Quantum Field Theory depend on the large energy cutoff, or equivalently, in position space, on a small distance cutoff. Low energy EFTs are used to describe a wide range of physics: from particle interactions to phenomenological models of the Universe.

However, for a low energy theory to be a complete full fundamental theory, one requires UV-completion, meaning that EFT will have constraints coming from the full UV- complete theory. One must determine the number of parameters from the small distance scale that are relevant at the large distance scale. There is also importance of the degrees of freedom from the underlying theory that appear at large distances (IR region).

Effective Field Theory is also an efficient way of characterizing new physics: it describes new physics in terms of coefficients of higher dimension operator, and includes contraints resulting from analyticity, locality, gauge and Lorentz invariance.

In this project I have reviewed the S- matrix program: requirement of analyticity (and causality), unitarity, locality on the scattering matrix $\hat{S}$. An example of Standard Model Effective Field Theory (SMEFT) resulting in flavour constraints of fermion scatterings was reviewed: for generalized elastic bounds any flavor-violating Wilson coefficients is constrained by the flavor-conserving coefficients.

Results from analyticity, unitarity and crossing symmetry were then viewed in the context of the 4-particle scattering amplitude, expressed as a function of the two Mandelstam variables $s = (p_1 + p_2)^2$ and $t = (p_1 - p_3)^2$, that satisfy the crossing symmetry relation, the optical theorem and the Froissart-Martin bound of the S-matrix program. I have reviewed top- bottom and bottom- up methods: first one is allowing one to move from UV- complete fundamental theory to a low energy approximation EFT, the latter one starts from experimentally verified observables and build respective EFT and examines whether the theory can be UV- completed.

The UV theory is assumed to satisfy the S-matrix axioms, which give rise to EFT constraints in the IR region: positivity bounds on Wilsonian coefficients. A positivity bound on the coupling c was rederived for scalar elastic forward 2-2 scattering. Thus, it was rederived how the dispersion relation arguments force the positivity condition: the scattering amplitude in the forward scattering limit $t = 0$ $A(s)$ at the tree level will include an $\sim s^2$ term with some coefficient in front, then this coefficient was shown to be strictly positive. These positivity bounds can be extended away from the forward scattering limit via Legendre Polynomial properties as described in [27],

[50]. Moreover, the properties of the partial wave expansion can also be used to derive an infinite series of non-forward scattering positivity bounds as in [51].

Another case briefly reviewed in this dissertation was for massive vector fields of the spin- 1, which is generally a more complicated case due to non- trivial scattering amplitude and the polarization vectors. I followed the steps described [42], where the bottom- up method is used for the EFT construction. One starts with a massless vector A_μ (Maxwell field) and then gets a massive field (Proca field) via Stückelberg mechanism. Then for a 2-2 vector field scattering, an EFT Lagrangian with only relevant terms was demonstated. Using the helicity and transversity formalism the scattering amplitudes were then computed. Using the positivity bound obtained from analyticity and unitarity requirements of $A(s, t, u)$, the corresponding positivity constraints on Wilson coefficients were then shown.

Although all the cases considered in this dissertation are at the tree- level, as a suggestion for further discussion, one could focus on computing scattering amplitudes beyond the tree- level: at loop Feynman diagrams, e.g. 4- Higgs interactions described in [22].

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